A super MHV vertex expansion for $\mathcal{N} = 4$ SYM theory

Michael Kiermaier[a] **and Stephen G. Naculich**[b]

[a] *Center for Theoretical Physics, Massachusetts Institute of Technology,*
77 Massachusetts Avenue, Cambridge, MA 02139, U.S.A.

[b] *Department of Physics, Bowdoin College,*
Brunswick, ME 04011, U.S.A.

E-mail: mkiermai@mit.edu, naculich@bowdoin.edu

ABSTRACT: We present a supersymmetric generalization of the MHV vertex expansion for all tree amplitudes in $\mathcal{N} = 4$ SYM theory. In addition to the choice of a reference spinor, this *super MHV vertex expansion* also depends on four reference Grassmann parameters. We demonstrate that a significant fraction of diagrams in the expansion vanishes for a judicious choice of these Grassmann parameters, which simplifies the computation of amplitudes. Even pure-gluon amplitudes require fewer diagrams than in the ordinary MHV vertex expansion.

We show that the super MHV vertex expansion arises from the recursion relation associated with a *holomorphic all-line supershift*. This is a supersymmetric generalization of the holomorphic all-line shift recently introduced in arXiv:0811.3624. We study the large-z behavior of generating functions under these all-line supershifts, and find that they generically provide $1/z^k$ falloff at (Next-to)^{k}MHV level. In the case of anti-MHV generating functions, we find that a careful choice of shift parameters guarantees a stronger $1/z^{k+4}$ falloff. These particular all-line supershifts may therefore play an important role in extending the super MHV vertex expansion to $\mathcal{N} = 8$ supergravity.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry

Contents

1 Introduction

On-shell tree-level scattering amplitudes exhibit a simplicity that is not at all evident from standard Feynman diagram calculations. This simplicity can be uncovered using recursion relations for on-shell amplitudes, of which the MHV vertex expansion [1] and the BCFW recursion relation [2, 3] are prominent examples. Recursion relations express an on-shell amplitude in terms of simpler on-shell amplitudes with fewer external legs, thus in principle allowing a recursive computation of arbitrarily complicated tree amplitudes.

The MHV vertex expansion expresses an amplitude as a sum over diagrams, each of which is a product of MHV amplitudes connected by scalar propagators. The individual diagrams depend on an arbitrarily-chosen reference spinor $|X]$, but the sum of diagrams is independent of $|X]$. The MHV vertex expansion is a convenient method for computing

amplitudes due to the simplicity of its basic building blocks, the MHV subamplitudes. The MHV vertex expansion reproduces the correct on-shell tree amplitudes both in pure Yang-Mills theory [4] and in $\mathcal{N} = 4$ SYM theory [5, 6].

Classes of amplitudes of $\mathcal{N} = 4$ SYM theory that are related by supersymmetric Ward identities [7–9] can be conveniently packaged into *generating functions* (also called superamplitudes). A generating function for n-point amplitudes depends not only on the momenta p_i of the external particles, but also on $4n$ Grassmann variables η_{ia}. (Here, $a = 1, 2, 3, 4$ is an SU(4) index.) Any particular amplitude can be obtained by acting on the generating function with a corresponding Grassmann differential operator [10]. The generating function for MHV amplitudes was first given in ref. [11]. Beyond the MHV level, a generating function for $\mathcal{N} = 4$ SYM amplitudes can be represented as a sum over terms involving dual superconformal invariants. At the NMHV level, this was carried out in ref. [12]. The generating function for all tree amplitudes of $\mathcal{N} = 4$ SYM theory was then obtained in ref. [13] by explicitly solving a supersymmetric generalization of the BCFW recursion relation [14–16]. Alternatively, a generating function beyond the MHV level can be written as a sum over the diagrams of the MHV vertex expansion. At the NMHV level, this form of the generating function was first presented in ref. [17]. For general tree amplitudes in $\mathcal{N} = 4$ SYM theory, an explicit representation of the generating function associated with the MHV vertex expansion was derived in ref. [6].

In this paper we present an alternative representation for $\mathcal{N} = 4$ SYM generating functions, based on a new recursion relation, the *super MHV vertex expansion*. The super MHV vertex expansion is a natural generalization of the ordinary MHV vertex expansion, and can be derived from a supersymmetry transformation acting on the latter. The diagrams of the super MHV vertex expansion depend not only on a reference spinor $|X]$, but also on an arbitrarily-chosen set of four reference Grassmann parameters η_{Xa}. The sum of diagrams is, of course, independent of these choices of reference parameters. With the choice $\eta_{Xa} = 0$, the super MHV vertex expansion reduces to the ordinary MHV vertex expansion, but for well-chosen values of η_{Xa}, the super MHV vertex expansion contains significantly fewer diagrams. It can thus be used to simplify the computation of both generic and pure-gluon amplitudes.

The computational simplification occurs because each of the four η_{Xa} can be chosen to eliminate all diagrams from the super MHV vertex expansion which contain an internal line with a particular momentum channel. As the same momentum channel generically occurs in many diagrams, and as four distinct channels can be eliminated, many diagrams can be made to vanish in this way. Surprisingly, even the computation of pure-gluon tree amplitudes, which coincide with gluon amplitudes of gauge theory with no supersymmetry, can be simplified using the super MHV vertex expansion. We illustrate the simplified computation of amplitudes and the counting of eliminated diagrams in examples.[1]

Recursion relations can be derived from the analytic behavior of an on-shell amplitude under a complex shift of its external momenta, and different shifts generically lead to differ-

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[1] For a quantitative comparison of the number of diagrams in the super MHV vertex expansion to the ordinary MHV vertex expansion, the reader is referred to table 1 in section 3.4.

ent recursion relations [3, 4]. Recursion relations were initially developed using shifts of a subset of the external momenta. However, the ordinary MHV vertex expansion most naturally follows from a holomorphic all-line shift, which deforms all the external momenta [6].

We show that the super MHV vertex expansion introduced in this paper follows naturally from the behavior of generating functions under a *holomorphic all-line supershift*. This supershift is a generalization of the holomorphic all-line shift presented in ref. [6], and it shifts not only the external momenta but also the Grassmann variables η_{ia} appearing in the generating functions. This is analogous to the generalization of the BCFW two-line shift to the two-line supershift recently introduced in ref. [14–16] and applied in ref. [13]. We prove that $\mathcal{N} = 4$ SYM generating functions beyond the MHV level vanish when the complex shift parameter z of the all-line supershift becomes large. This supershift therefore yields a valid recursion relation, which in turn generates the super MHV vertex expansion. As in the super BCFW recursion relation, which follows from the two-line supershift, the diagrams in the super MHV vertex expansion of an amplitude do not have an immediate interpretation as products of ordinary tree amplitudes. Specific amplitudes, however, are readily computed directly from the generating function.

We examine the behavior of anti-MHV generating functions under holomorphic all-line supershifts in detail, and find that the falloff at large z is faster when the choices of reference spinor $|X]$ and Grassmann parameters η_{Xa} are correlated in a certain way. The analysis of this special case is motivated by a possible future application of all-line supershift recursion relations to $\mathcal{N} = 8$ supergravity. In fact, as we will argue, the extra suppression implies that a super MHV vertex expansion for $\mathcal{N} = 8$ supergravity must exist at least for all anti-MHV amplitudes.

This paper is organized as follows. In section 2, we review MHV amplitudes and the concept of generating functions in $\mathcal{N} = 4$ SYM theory. In section 3, we present the super MHV vertex expansion as a generalization of the ordinary MHV vertex expansion. We discuss its properties and demonstrate how it can simplify the computation of amplitudes. We introduce holomorphic all-line supershifts in section 4. We study the large z behavior of generating functions under these shifts both for generic and special choices of shift parameters. In section 5, we show that the super MHV vertex expansion arises from the recursion relations associated with holomorphic all-line supershifts. Finally, in section 6, we discuss the relation of our work to other recent developments. We also comment on the prospects of generalizing the super MHV vertex expansion to $\mathcal{N} = 8$ supergravity.

2 Review

The simplest on-shell amplitudes in $\mathcal{N} = 4$ SYM theory are n-point MHV amplitudes. The MHV sector contains amplitudes with negative helicity gluons on two lines and positive helicity gluons on the remaining lines, together with all amplitudes related to these by supersymmetry. An n-point MHV amplitude takes the simple form[2]

$$\mathcal{A}_n^{\mathrm{MHV}}(1,\ldots,n) \;=\; \frac{\langle \ldots \rangle \langle \ldots \rangle \langle \ldots \rangle \langle \ldots \rangle}{\mathrm{cyc}(1,\ldots,n)}\,, \quad \mathrm{with} \quad \mathrm{cyc}(1,\ldots,n) = \prod_{i=1}^{n} \langle i, i+1 \rangle\,. \quad (2.1)$$

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[2]Throughout this paper, we use the spinor-helicity formalism, with the conventions summarized in appendix A of ref. [10].

The four angle brackets $\langle\ldots\rangle$ in the numerator depend on the choice of states on the external lines $1,\ldots,n$. For the pure-gluon MHV amplitude with negative helicity gluons on lines i and j, for example, the numerator takes the form $\langle ij\rangle^4$, and we obtain the well-known Parke-Taylor formula [18]:

$$\mathcal{A}_n^{\mathrm{MHV}}(\ldots i^-\ldots j^-\ldots) \;=\; \frac{\langle ij\rangle^4}{\mathrm{cyc}(1,\ldots,n)}\,. \tag{2.2}$$

All n-point MHV amplitudes can be conveniently encapsulated in the n-point MHV generating function (or superamplitude) [11]

$$\mathcal{F}_n^{\mathrm{MHV}}(\eta_{ia}) = \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(1,\ldots,n)} \quad\text{with}\quad \delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_{ia}\right) = \frac{1}{2^4}\prod_{a=1}^4\sum_{i,j=1}^n \langle ij\rangle\eta_{ia}\eta_{ja}\,. \tag{2.3}$$

Specific amplitudes may then be extracted from $\mathcal{F}_n^{\mathrm{MHV}}(\eta_{ia})$ by acting with an eighth-order Grassmann differential operator $D^{(8)}$ built from operators associated with the external states of the amplitude:

$$\mathcal{A}_n^{\mathrm{MHV}}(1,\ldots,n) = D^{(8)}\mathcal{F}_n^{\mathrm{MHV}}(\eta_{ia})\,. \tag{2.4}$$

The Grassmann differential operators associated with a particular choice of external particle on line i, in order of increasing helicity, are given by [10]:

$$D_i^- = \frac{\partial^4}{\partial\eta_{i1}\partial\eta_{i2}\partial\eta_{i3}\eta_{i4}}\,,\quad D_i^{abc} = \frac{\partial^3}{\partial\eta_{ia}\partial\eta_{ib}\partial\eta_{ic}}\,,\quad D_i^{ab} = \frac{\partial^2}{\partial\eta_{ia}\partial\eta_{ib}}\,,\quad D_i^a = \frac{\partial}{\partial\eta_{ia}}\,,\quad D_i^+ = 1\,. \tag{2.5}$$

The indices $a,b,c = 1,\ldots,4$ on the operators D_i^{abc}, D_i^{ab}, and D_i^a associated with scalars and gluinos are SU(4) indices, and distinct choices of SU(4) indices correspond to distinct choices of scalars and gluinos. The pure-gluon MHV amplitude with negative helicity gluons on lines i and j given in eq. (2.2) is now readily obtained from $\mathcal{F}_n^{\mathrm{MHV}}(\eta_{ia})$ as

$$D^{(8)} \;=\; D_i^- D_j^- \;\implies\; \mathcal{A}_n^{\mathrm{MHV}}(\ldots i^-\ldots j^-\ldots) \;=\; D_i^- D_j^- \mathcal{F}_n^{\mathrm{MHV}}(\eta_{ia}) \;=\; \frac{\langle ij\rangle^4}{\mathrm{cyc}(1,\ldots,n)}\,. \tag{2.6}$$

All non-MHV n-point amplitudes may be classified as $(\text{Next-to})^k$MHV amplitudes for integer k between 1 and $n-4$.[3] The N^kMHV sector contains amplitudes with $k+2$ negative helicity gluons and $n-k-2$ positive helicity gluons, together with all amplitudes related to these by supersymmetry. All N^kMHV amplitudes may also be packaged into generating functions $\mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}}(\eta_{ia})$, from which a specific amplitude $\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}$ may be extracted through

$$\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}(1,\ldots,n) = D^{(8+4k)}\mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}}(\eta_{ia})\,. \tag{2.7}$$

The order $8+4k$ Grassmann differential operator $D^{(8+4k)}$ is built from a product of operators in (2.5) associated with the states on each external line i.

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[3]The maximal value $k = n-4$ corresponds to anti-MHV amplitudes, which can be expressed as in eq. (2.1), but with angle brackets replaced by square brackets.

3 The super MHV vertex expansion

In this section, we explain the super MHV vertex expansion and explore its consequences. In section 3.1, we review the ordinary MHV vertex expansion for NMHV amplitudes and present the super MHV vertex expansion as its generalization. We also show that the sum rule recently found in ref. [6] is an immediate consequence of this expansion. We demonstrate in section 3.2 that the reference Grassmann parameters of the super MHV vertex expansion can be chosen so as to simplify NMHV amplitude computations. In section 3.3 we generalize the super MHV vertex expansion to all tree amplitudes in $\mathcal{N} = 4$ SYM theory, and in section 3.4 we analyze the computational simplifications for general tree amplitudes.

3.1 NMHV amplitudes

The ordinary MHV vertex expansion [1] gives a simple prescription to compute arbitrary tree-level amplitudes in $\mathcal{N} = 4$ SYM theory. At the NMHV level, for example, one is instructed to sum over all possible diagrams in which an n-point NMHV amplitude $\mathcal{A}_n^{\mathrm{NMHV}}$ can be split into two MHV subamplitudes I_1 and I_2, connected by an internal line of momentum P_α (see figure 1a). Each diagram is characterized by the subset α of external lines whose momenta flow into the internal line, i.e.

$$P_\alpha = \sum_{i \in \alpha} p_i \,. \tag{3.1}$$

In figure 1a, the set α thus consists of all external lines on subamplitude I_1.[4] The sum over diagrams gives the desired NMHV amplitude:

$$\mathcal{A}_n^{\mathrm{NMHV}}(1, \ldots, n) = \sum_{\text{diagrams } \alpha} \mathcal{A}^{\mathrm{MHV}}(I_1) \frac{1}{P_\alpha^2} \mathcal{A}^{\mathrm{MHV}}(I_2) \,. \tag{3.2}$$

The momentum P_α of the internal line is not null, and we thus need to explain how to treat the angle brackets $|P_\alpha\rangle$ which are needed to compute the MHV subamplitudes in eq. (3.2). The CSW prescription [1] instructs us to use

$$|P_\alpha\rangle \equiv P_\alpha|X] \,, \tag{3.3}$$

where $|X]$ is an arbitrarily chosen reference spinor. Each diagram in eq. (3.2) will generically depend on the choice of $|X]$, but their sum is guaranteed to reproduce the correct amplitude independently of the chosen reference spinor [4, 5].

All n-point NMHV amplitudes can be packaged into an NMHV generating function, in terms of which the MHV vertex expansion (3.2) may be written as [17]

$$\mathcal{F}_n^{\mathrm{NMHV}}(\eta_{ia}) = \sum_{\text{diagrams } \alpha} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle \eta_{ia}\big)}{\mathrm{cyc}(I_1) \, P_\alpha^2 \, \mathrm{cyc}(I_2)} \prod_{a=1}^4 \sum_{i \in \alpha} \langle iP_\alpha \rangle \eta_{ia} \,. \tag{3.4}$$

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[4]Note that all external momenta p_i are outgoing in our conventions, which explains the direction of the arrow on the internal line P_α.
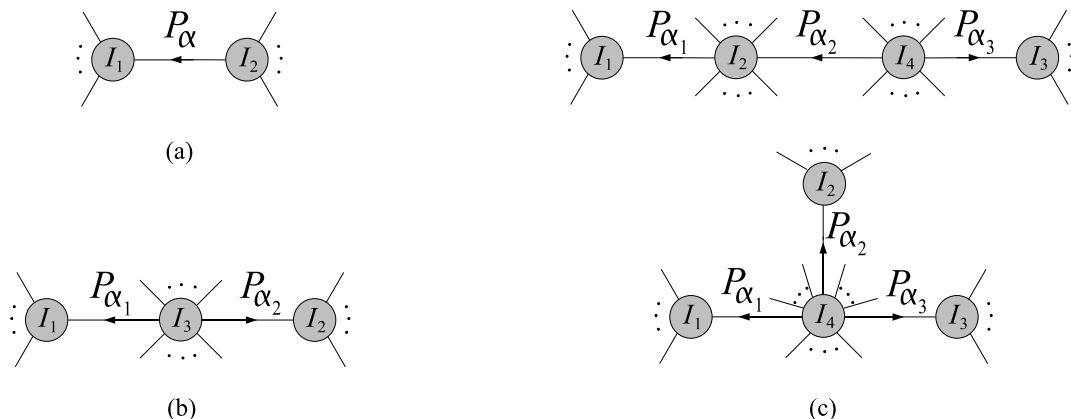
Figure 1. The diagrams of the MHV vertex expansion at the (a) NMHV, (b) N^2MHV, and (c) N^3MHV level.

Any particular NMHV amplitude can be obtained from this expression by using eq. (2.7) with $k = 1$.

To define a supersymmetric generalization of eq. (3.4), we first recall that supercharges $[Q^a|$ and $|\tilde{Q}_a\rangle$ were defined in ref. [10] as

$$|\tilde{Q}_a\rangle \equiv \sum_{i=1}^{n} |i\rangle \, \eta_{ia} \,, \qquad\qquad [Q^a| \equiv \sum_{i=1}^{n} [i| \, \frac{\partial}{\partial \eta_{ia}} \,. \qquad (3.5)$$

On a function $f(\eta_{ia})$, $[Q^a|$ generates the SUSY transformation

$$f(\eta_{ia}) \;\longrightarrow\; \tilde{f}(\eta_{ia}) \;\equiv\; \exp\big([Q^a \, \epsilon_a]\big) f(\eta_{ia}) \;=\; f\big(\eta_{ia} + [\epsilon_a \, i]\big) \qquad (3.6)$$

and in particular

$$\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}\big(\eta_{ia}\big) \equiv \exp\big([Q^a \, \epsilon_a]\big) \, \mathcal{F}_n^{\mathrm{NMHV}}\big(\eta_{ia}\big) = \mathcal{F}_n^{\mathrm{NMHV}}\big(\eta_{ia} + [\epsilon_a \, i]\big) \,. \qquad (3.7)$$

But since the generating function (3.4) encodes on-shell amplitudes, it must be SUSY invariant:[5]

$$[Q^a \, \epsilon_a] \, \mathcal{F}_n^{\mathrm{NMHV}}(\eta_{ia}) = 0 \,. \qquad (3.8)$$

Therefore, although each individual diagram in (3.4) transforms non-trivially under supersymmetry, the function $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia})$ is equal to $\mathcal{F}_n^{\mathrm{NMHV}}(\eta_{ia})$ after summing over diagrams. Thus $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia})$ is a valid generating function of NMHV amplitudes in $\mathcal{N} = 4$ SYM theory.

Let us now choose

$$[\epsilon_a| = [Y|\eta_{Xa} \quad \text{ with } \quad [XY] = 1 \,, \qquad (3.9)$$

where we picked the normalization condition for later convenience. Then $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$, defined through eqs. (3.7) and (3.9), is the *super MHV vertex expansion* of the NMHV

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[5]For the NMHV generating function in the representation (3.4), this was confirmed explicitly in ref. [10].

generating function. Using

$$\sum_{i \in \alpha} \langle iP_\alpha \rangle \Big(\eta_{ia} + [Yi]\eta_{Xa} \Big) = \sum_{i \in \alpha} \Big(\langle iP_\alpha \rangle \eta_{ia} + [Yi]\langle i|P_\alpha|X]\eta_{Xa} \Big)$$
$$= \sum_{i \in \alpha} \langle iP_\alpha \rangle \eta_{ia} + P_\alpha^2 \, \eta_{Xa} \,, \qquad (3.10)$$

the super MHV vertex expansion takes the simple form[6]

$$\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa}) \; = \; \sum_{\mathrm{diagrams} \; \alpha} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle \eta_{ia}\big)}{\mathrm{cyc}(I_1) \, P_\alpha^2 \, \mathrm{cyc}(I_2)} \; \prod_{a=1}^4 \Bigg[\sum_{i \in \alpha} \langle iP_\alpha \rangle \eta_{ia} + P_\alpha^2 \, \eta_{Xa} \Bigg] . \quad (3.11)$$

The individual terms of eq. (3.11) depend both on a choice of reference spinor $|X]$ and on a choice of *four reference Grassmann parameters* η_{Xa}. For any choice of $|X]$ and η_{Xa}, $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$ is the generating function of NMHV amplitudes, which are obtained, as in eq. (2.7), by applying the twelfth-order Grassmann differential operator $D^{(12)}$ associated with the external states of the amplitude:

$$\mathcal{A}_n^{\mathrm{NMHV}}(1, \ldots, n) = D^{(12)} \tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa}) \,. \qquad (3.12)$$

If the Grassmann differential operators in $D^{(12)}$ do not act on η_{Xa}, eq. (3.12) reproduces the ordinary MHV vertex expansion. However, if we choose each η_{Xa} to be a linear combination of the Grassmann variables $\eta_{1a}, \ldots, \eta_{na}$ associated with external states, then the operators in $D^{(12)}$ do indeed act on η_{Xa}. With such a choice, $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$ is, diagram by diagram, a sum of twelfth-order monomials in the Grassmann variables η_{ia}, each monomial containing three powers of η_{ia} for each fixed SU(4) index $a = 1, 2, 3, 4$. While eq. (3.8) guarantees the equivalence of $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$ and $\mathcal{F}_n^{\mathrm{NMHV}}(\eta_{ia})$, they lead to distinct diagrammatic expansions for amplitudes. Therefore the super MHV vertex expansion is a non-trivial generalization of the ordinary MHV vertex expansion when each η_{Xa} is chosen as some linear combination of the external η_{ia}.

As $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$ is independent of η_{Xa} after summing over diagrams, its derivative with respect to each η_{Xa} must vanish. We can thus derive sum rules from eq. (3.11) by differentiation. A particularly interesting sum rule can be obtained by differentiating off the entire dependence of $\tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa})$ on η_{Xa}, yielding

$$0 = \left[\prod_{a=1}^4 \frac{\partial}{\partial \eta_{Xa}} \right] \tilde{\mathcal{F}}_n^{\mathrm{NMHV}}(\eta_{ia}; \eta_{Xa}) = \sum_{\mathrm{diagrams} \; \alpha} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle \eta_{ia}\big)}{\mathrm{cyc}(I_1) \, P_\alpha^2 \, \mathrm{cyc}(I_2)} \, \big(P_\alpha^2\big)^4 \,. \qquad (3.13)$$

Pulling out an overall diagram-independent factor of $\delta^{(8)}(\sum_i |i\rangle \eta_{ia})/\mathrm{cyc}(1, \ldots, n)$, we obtain

$$0 \; = \; \sum_{\mathrm{diagrams} \; \alpha} W_\alpha \big(P_\alpha^2\big)^4 \quad \mathrm{with} \quad W_\alpha \equiv \frac{\mathrm{cyc}(1, \ldots, n)}{\mathrm{cyc}(I_1) \, P_\alpha^2 \, \mathrm{cyc}(I_2)} \,. \qquad (3.14)$$

This is the main sum rule derived in section 8.4 of ref. [6], which now finds a natural interpretation as an immediate consequence of the η_{Xa} independence of the super MHV vertex expansion.

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[6] For $\eta_{Xa} = 0$, the super MHV vertex expansion (3.11) manifestly reduces to the ordinary MHV vertex expansion (3.4).

3.2 Simplified NMHV amplitude computations

At first sight, the η_{Xa} dependence of the super MHV vertex expansion (3.11) may seem like an unnecessary complication. This would suggest that the choice $\eta_{Xa} = 0$, which reduces it to the ordinary MHV vertex expansion, is most convenient. However, inspection of eq. (3.11) shows that we can choose the η_{Xa} in such a way that certain diagrams in the generating function vanish identically. For example, pick any *four* diagrams in the MHV vertex expansion and denote them by $\beta_1, \beta_2, \beta_3,$ and β_4. By choosing

$$\eta_{Xa} = -\frac{1}{P_{\beta_a}^2} \sum_{i \in \beta_a} \langle iP_{\beta_a} \rangle \eta_{ia} \,, \tag{3.15}$$

we guarantee that the four diagrams β_a no longer contribute to the sum over α in the generating function $\tilde{\mathcal{F}}_n^{\text{NMHV}}(\eta_{ia}; \eta_{Xa})$. Note that this implies that these four diagrams do not contribute to *any* amplitude $\mathcal{A}_n^{\text{NMHV}}$ computed from eq. (3.11). We have chosen each η_{Xa} as a different linear combination of the η_{ia}, to maximize the simplification. We obtain

$$\tilde{\mathcal{F}}_n^{\text{NMHV}}(\eta_{ia}; \eta_{Xa}) = \sum_{\substack{\text{diagrams} \\ \alpha \neq \beta_1, \beta_2, \beta_3, \beta_4}} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle \eta_{ia}\big)}{\text{cyc}(I_1) \, P_\alpha^2 \, \text{cyc}(I_2)} \prod_{a=1}^4 \left[\sum_{i \in \alpha} \langle iP_\alpha \rangle \eta_{ia} - \frac{P_\alpha^2}{P_{\beta_a}^2} \sum_{i \in \beta_a} \langle iP_{\beta_a} \rangle \eta_{ia} \right]. \tag{3.16}$$

Here, we have explicitly excluded the diagrams $\beta_1, \ldots, \beta_4$ in the sum, as diagrams with $\alpha = \beta_a$ involve a vanishing factor in the product over a and therefore do not contribute. At the NMHV level, the super MHV vertex expansion thus allows us to eliminate four diagrams. As we will see below, many more diagrams can be eliminated at higher N^kMHV level. The simplification thus grows with increasing level in k. For now, let us illustrate the power of the super MHV vertex expansion in the form (3.16) with an example.

Example: NMHV 5-point amplitudes. As the simplest example, let us consider the NMHV amplitude with $n = 5$ external lines. (Five-point NMHV amplitudes can be treated as anti-MHV, which provides a consistency check on our calculation. Note that it is our goal to illustrate the advantages of the super MHV vertex expansion over the ordinary MHV vertex expansion with this example; we do not expect to achieve simplifications compared to the trivial anti-MHV computation.) Five diagrams contribute to the ordinary MHV vertex expansion for the NMHV 5-point amplitude (see figure 2a). With the choice (3.15) for η_{Xa}, we can eliminate four of these from the super MHV vertex expansion. For definiteness, we choose

$$\beta_1 = \{1, 2, 3\}, \qquad \beta_2 = \{2, 3\}, \qquad \beta_3 = \{5, 1\}, \qquad \beta_4 = \{5, 1, 2\}. \tag{3.17}$$

The unique diagram which then contributes to $\tilde{\mathcal{F}}_5^{\text{NMHV}}(\eta_{ia}; \eta_{Xa})$ is the diagram $\alpha = \{1, 2\}$ (see figure 1b). Explicitly, eq. (3.16) gives

$$\tilde{\mathcal{F}}_5^{\text{NMHV}} = \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle \eta_{ia}\big) \prod_{a=1}^4 \Big[\langle 1P_{12} \rangle \eta_{1a} + \langle 2P_{12} \rangle \eta_{2a} - \big(P_{12}^2/P_{\beta_a}^2\big) \sum_{k \in \beta_a} \langle kP_{\beta_a} \rangle \eta_{ka} \Big]}{\langle 12 \rangle \langle 2P_{12} \rangle \langle P_{12}1 \rangle \, P_{12}^2 \, \langle 34 \rangle \langle 45 \rangle \langle 5P_{12} \rangle \langle P_{12}3 \rangle} \,. \tag{3.18}$$
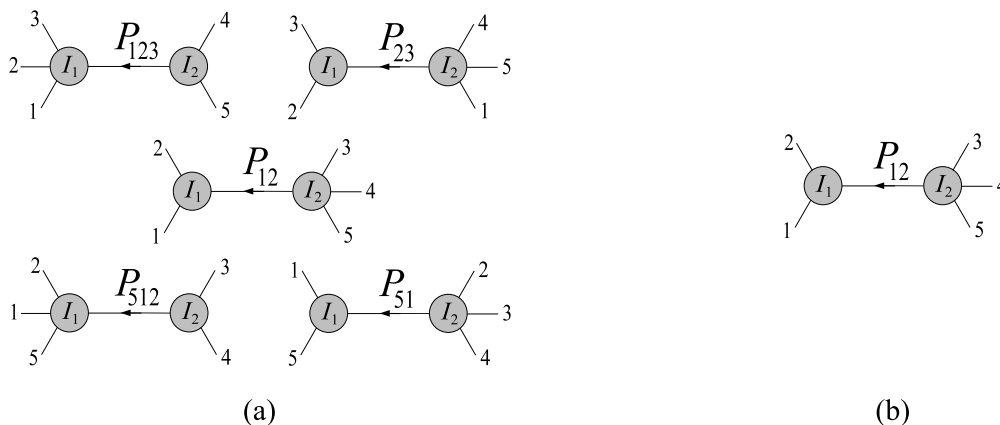
Figure 2. (a) The five diagrams contributing to the ordinary MHV vertex expansion of 5-point NMHV amplitudes. (b) The single remaining diagram that contributes to the super MHV vertex expansion.

To test this generating function, let us compute the gluon amplitude $\mathcal{A}_5(1^+, 2^+, 3^-, 4^-, 5^-)$. Since this amplitude is anti-MHV, the conjugate of the Parke-Taylor formula immediately gives

$$\mathcal{A}_5(1^+, 2^+, 3^-, 4^-, 5^-) = \frac{[12]^4}{[12][23][34][45][51]}. \tag{3.19}$$

Computing this amplitude with the ordinary MHV vertex expansion, however, is messy and the simple result (3.19) is difficult to obtain analytically. Four diagrams contribute to its expansion; in fact, precisely the four diagrams $\beta_1, \ldots, \beta_4$ listed in eq. (3.17). In the super MHV vertex expansion the computation simplifies dramatically. Acting with $D_3^- D_4^- D_5^-$ on eq. (3.18), we find

$$D_3^- D_4^- D_5^- \tilde{\mathcal{F}}_5^{\mathrm{NMHV}}$$
$$= \frac{\prod_{a=1}^4 \frac{\partial}{\partial \eta_{3a}} \frac{\partial}{\partial \eta_{4a}} \frac{\partial}{\partial \eta_{5a}} \left[\frac{1}{2} \sum_{i,j=1}^n \langle ij \rangle \eta_{ia} \eta_{ja} \right] \left[\langle 1P_{12} \rangle \eta_{1a} + \langle 2P_{12} \rangle \eta_{2a} - \left(P_{12}^2/P_{\beta_a}^2 \right) \sum_{k \in \beta_a} \langle kP_{\beta_a} \rangle \eta_{ka} \right]}{\langle 12 \rangle \langle 2P_{12} \rangle \langle P_{12}1 \rangle \ P_{12}^2 \ \langle 34 \rangle \langle 45 \rangle \langle 5P_{12} \rangle \langle P_{12}3 \rangle}. \tag{3.20}$$

Consider first the factors $a = 1$ and $a = 2$. In both cases, the derivatives with respect to η_{4a} and η_{5a} must act on the first factor, and we obtain

$$\frac{\partial}{\partial \eta_{3a}} \frac{\partial}{\partial \eta_{4a}} \frac{\partial}{\partial \eta_{5a}} \left[\frac{1}{2} \sum_{i,j=1}^n \langle ij \rangle \eta_{ia} \eta_{ja} \right] \left[\langle 1P_{12} \rangle \eta_{1a} + \langle 2P_{12} \rangle \eta_{2a} - \left(P_{12}^2/P_{\beta_a}^2 \right) \sum_{k \in \beta_a} \langle kP_{\beta_a} \rangle \eta_{ka} \right]$$

$$\longrightarrow \ \langle 45 \rangle \frac{P_{12}^2}{P_{123}^2} \langle 3P_{123} \rangle = \frac{[12] \langle 12 \rangle \langle 3P_{12} \rangle}{[45]} \qquad \text{for } a = 1,$$

$$\longrightarrow \ \langle 45 \rangle \frac{P_{12}^2}{P_{23}^2} \langle 3P_{23} \rangle = -\frac{\langle 45 \rangle [12] \langle 12 \rangle [2X]}{[23]} = -\frac{\langle 45 \rangle [12] \langle 1P_{12} \rangle}{[23]} \qquad \text{for } a = 2. \tag{3.21}$$

The remaining cases $a = 3$ and $a = 4$ follow by relabeling from the cases $a = 2$ and $a = 1$, respectively. We obtain

$$
\begin{aligned}
&D_3^- D_4^- D_5^- \tilde{\mathcal{F}}_5^{\text{NMHV}} \\
&= \frac{\big([12]\langle 12\rangle\langle 3P_{12}\rangle/[45]\big) \cdot \big(-\langle 45\rangle[12]\langle 1P_{12}\rangle/[23]\big) \cdot \big(-\langle 34\rangle[12]\langle 2P_{12}\rangle/[51]\big) \cdot \big([12]\langle 12\rangle\langle 5P_{12}\rangle/[34]\big)}{\langle 12\rangle\langle 2P_{12}\rangle\langle P_{12}1\rangle \; P_{12}^2 \; \langle 34\rangle\langle 45\rangle\langle 5P_{12}\rangle\langle P_{12}3\rangle} \\
&= \frac{[12]^4}{[12][23][34][45][51]} \, .
\end{aligned}
\tag{3.22}
$$

We have thus reproduced the simple anti-MHV result (3.19).

The simplicity of this computation was not just a consequence of our particular choice of external states. In fact, the generating function $\tilde{\mathcal{F}}_5^{\text{NMHV}}(\eta_{ia}; \eta_{Xa})$ can be manipulated to explicitly obtain the anti-MHV generating function. For example, the $a = 2$ factor in eq. (3.18) gives

$$
\begin{aligned}
&\langle 1P_{12}\rangle\eta_{1a} + \langle 2P_{12}\rangle\eta_{2a} - \frac{P_{12}^2}{P_{23}^2}\Big(\langle 2P_{23}\rangle\eta_{2a} + \langle 3P_{23}\rangle\eta_{3a}\Big) \\
&\qquad = \frac{1}{[23]}\Big([23]\langle 1P_{12}\rangle\eta_{1a} - \langle 12\rangle\big([23][1X] + [12][3X]\big)\eta_{2a} + [12]\langle 12\rangle[2X]\eta_{3a}\Big) \\
&\qquad = \frac{\langle 1P_{12}\rangle}{[23]}\Big([23]\eta_{1a} + [31]\eta_{2a} + [12]\eta_{3a}\Big),
\end{aligned}
\tag{3.23}
$$

where we used the Schouten identity $|1][23] + \text{cyclic} = 0$. The other terms can be treated analogously. The resulting prefactors $\langle iP_{12}\rangle$ cancel the four $|X]$-dependent angle brackets in the denominator of eq. (3.18), and we obtain

$$
\tilde{\mathcal{F}}_5^{\text{NMHV}} = \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\langle 45\rangle^2\langle 34\rangle^2 \prod_{i=1}^5 [i, i+1]} \prod_{a=1}^2\Big([23]\eta_{1a} + [31]\eta_{2a} + [12]\eta_{3a}\Big)\prod_{a=3}^4\Big([25]\eta_{1a} + [51]\eta_{2a} + [12]\eta_{5a}\Big),
\tag{3.24}
$$

which is equivalent to the anti-MHV generating function for the 5-point amplitude presented in [5, 19]

$$
\mathcal{F}_5^{\overline{\text{MHV}}} = \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\langle 12\rangle^4 \prod_{i=1}^5 [i, i+1]} \prod_{a=1}^4\Big([34]\eta_{5a} + [45]\eta_{3a} + [53]\eta_{4a}\Big).
\tag{3.25}
$$

To see this equivalence, we notice that the lines 1 and 2 that are arbitrarily singled out in eq. (3.25) can in principle be chosen differently for each value of a. In eq. (3.24), lines 4 and 5 are singled out for $a = 1, 2$, and lines 3 and 4 are singled out for $a = 3, 4$.

NMHV amplitudes with $n \geq 6$ external states. For $n \geq 6$ external states, more than one diagram contributes to the super MHV vertex expansion of the NMHV generating function. For $n = 6$ ($n = 7$) there are a total of 9 diagrams (14 diagrams) in the ordinary MHV vertex expansion. As four of these diagrams can be made to vanish by choosing η_{Xa} as in eq. (3.15), the super MHV vertex expansion eliminates almost one-half (one-third) of the diagrams. Many more diagrams can be eliminated for N^kMHV amplitudes with $k > 1$, and we proceed to this case now.

3.3 All tree amplitudes

For a general n-point (Next-to)^{k}MHV tree amplitude $\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}$, the ordinary MHV vertex expansion instructs us [1] to sum over all possible diagrams in which the amplitude can be split into $k+1$ MHV subamplitudes $I_1, I_2,\ldots, I_{k+1}$, connected by k internal lines of momenta $P_{\alpha_1},\ldots, P_{\alpha_k}$. (See figures 1b and 1c for the types of MHV vertex diagrams which can occur at N^{2}MHV and N^{3}MHV level, respectively.) Each diagram is characterized by the subsets $\alpha_1,\ldots,\alpha_k$ of external lines whose momenta flow into the internal lines, i.e.

$$P_{\alpha_A} = \sum_{i\in\alpha_A} p_i \,. \tag{3.26}$$

The sum over all possible such MHV vertex diagrams gives the desired N^{k}MHV amplitude:

$$\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}(1,\ldots,n) = \sum_{\substack{\mathrm{MHV\ diagrams}\\ \{\alpha_1,\ldots,\alpha_k\}}} \frac{\mathcal{A}^{\mathrm{MHV}}(I_1)\cdots\mathcal{A}^{\mathrm{MHV}}(I_{k+1})}{P_{\alpha_1}^2\cdots P_{\alpha_k}^2}\,, \tag{3.27}$$

with the CSW prescription understood for each occurrence of the angle spinors $|P_{\alpha_A}\rangle$ in eq. (3.27):

$$|P_{\alpha_A}\rangle \equiv P_{\alpha_A}|X]\,. \tag{3.28}$$

The generating function associated with the ordinary MHV vertex expansion is [6]

$$\mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}}(\eta_{ia}) = \sum_{\substack{\mathrm{MHV\ diagrams}\\ \{\alpha_1,\ldots,\alpha_k\}}} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\cdots\mathrm{cyc}(I_{k+1})} \prod_{A=1}^k \left[\frac{1}{P_{\alpha_A}^2} \prod_{a=1}^4 \sum_{i\in\alpha_A} \langle iP_{\alpha_A}\rangle\eta_{ia} \right]. \tag{3.29}$$

To obtain the super MHV vertex expansion, we act with a SUSY transformation on the generating function (3.29):

$$\tilde{\mathcal{F}}_n^{\mathrm{N}^k\mathrm{MHV}}\big(\eta_{ia}\big) = \exp\left([Q^a \,\epsilon_a]\right) \mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}}\big(\eta_{ia}\big) = \mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}}\big(\eta_{ia} + [\epsilon_a\, i]\big)\,. \tag{3.30}$$

For the SUSY parameter, we again choose the $[\epsilon_a|$ defined in eq. (3.9). The super MHV vertex expansion for the N^{k}MHV generating function then takes the simple form

$$\tilde{\mathcal{F}}_n^{\mathrm{N}^k\mathrm{MHV}}(\eta_{ia}; \eta_{Xa}) = \sum_{\substack{\mathrm{MHV\ diagrams}\\ \{\alpha_1,\ldots,\alpha_k\}}} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\cdots\mathrm{cyc}(I_{k+1})} \prod_{A=1}^k \frac{1}{P_{\alpha_A}^2} \prod_{a=1}^4 \left[\sum_{i\in\alpha_A} \langle iP_{\alpha_A}\rangle\eta_{ia} + P_{\alpha_A}^2 \eta_{Xa} \right]. \tag{3.31}$$

Acting on $\tilde{\mathcal{F}}_n^{\mathrm{N}^k\mathrm{MHV}}(\eta_{ia}; \eta_{Xa})$ with the order $8+4k$ Grassmann differential operator $D^{(8+4k)}$ associated with an amplitude $\mathcal{A}_n^{\mathrm{N}^k\mathrm{MHV}}$, we obtain, diagram by diagram, the super MHV vertex expansion for that amplitude.

3.4 Simplification of general amplitude computations

To simplify general amplitude calculations, we choose the same strategy as in section 3.2 above. By picking η_{Xa} as in eq. (3.15) for some choice of channels $\beta_1, \beta_2, \beta_3$, and β_4, *all*

MHV vertex diagrams for which any internal line α_A coincides with any β_a vanish:

$$\alpha_A = \beta_a \quad \text{for } any \quad A = 1, \ldots, k, \quad \text{and } any \quad a = 1, \ldots, 4 \quad \implies \quad \text{diagram vanishes}.$$

(3.32)

For generic amplitude computations[7] the super MHV vertex expansion is most efficient if we maximize the number of diagrams which vanish by eq. (3.32). Different choices of β_a can lead to a different number of vanishing diagrams. Two simple guiding principles should be used for the choice of β_a:

1. **The channels β_a should appear in as many MHV vertex diagrams as possible.**

   Consider 10-point N^{3}MHV amplitudes. The channel $\alpha = \{1, 2, 3, 4, 5\}$ occurs in 123 distinct MHV vertex diagrams, while the channel $\alpha = \{1, 2\}$ occurs in 225 different diagrams. The choice $\beta_1 = \{1, 2\}$ is thus more efficient because it eliminates 102 more diagrams than $\beta_1 = \{1, 2, 3, 4, 5\}$.

2. **The channels β_a should, as far as possible, not occur in the same MHV vertex diagrams.**

   In the 10-point N^{3}MHV example, $\beta_1 = \{1, 2\}$ and $\beta_2 = \{2, 3\}$ cannot occur together in any MHV vertex diagram. The channels $\beta_1 = \{1, 2\}$ and $\beta_2 = \{3, 4\}$, on the other hand, appear together as internal lines in 20 different MHV vertex diagrams. In this case, β_2 eliminates 20 diagrams that were already eliminated by β_1. The total number of eliminated diagrams is thus reduced by 20 as compared to the choice $\beta_1 = \{1, 2\}$, $\beta_2 = \{2, 3\}$.

   Whether two channels β_1, β_2 can occur together in an MHV vertex diagram can be easily tested. If, possibly after using the freedom to relabel $\beta_1 \leftrightarrow \bar{\beta}_1$ and $\beta_2 \leftrightarrow \bar{\beta}_2$, the sets β_1 and β_2 do not share any external lines ($\beta_1 \cap \beta_2 = \emptyset$), then they can appear as internal lines in the same MHV vertex diagram.[8]

In table 1, we summarize the number of diagrams eliminated from the generating function for various choices of n and k. For simplicity we always used the choice

$$\beta_1 = \{1, 2\}, \qquad \beta_2 = \{2, 3\}, \qquad \beta_3 = \{3, 4\}, \qquad \beta_4 = \{4, 5\}$$

(3.33)

for the counting of super MHV vertex diagrams in table 1.

The number of diagrams of the ordinary vertex expansion at n-point N^{k}MHV level is given [20] by the expression $M(n, k) = \frac{1}{k+1}\binom{n-3}{k}\binom{n+k-1}{k}$. The number of diagrams of the super MHV vertex expansion for the choice (3.33) is then given by the expression[9]

$$S(n, k) = M(n, k) - 4M(n-1, k-1) + 3M(n-2, k-2).$$

(3.34)

---

[7]If we want to use the super MHV vertex expansion to compute only one specific amplitude, or a specific class of amplitudes (such as pure-gluon amplitudes), a different strategy is advisable. See discussion below.

[8]Here, $\bar{\beta}_a$ denotes the complement of the set β_a, regarded as a subset of all external lines $\{1, \ldots, n\}$.

[9]We thank Marcus Spradlin for a question which prompted us to derive this expression after v1 of this paper was submitted.

	$k=1$		$k=2$		$k=3$		$k=4$		$k=5$	
$n=5$	80%	ordinary: 5 super: 1								
$n=6$	44%	ordinary: 9 super: 5	81%	ordinary: 21 super: 4						
$n=7$	29%	ordinary: 14 super: 10	59%	ordinary: 56 super: 23	82%	ordinary: 84 super: 15				
$n=8$	20%	ordinary: 20 super: 16	44%	ordinary: 120 super: 67	66%	ordinary: 300 super: 103	83%	ordinary: 330 super: 57		
$n=9$	15%	ordinary: 27 super: 23	34%	ordinary: 225 super: 148	53%	ordinary: 825 super: 387	69%	ordinary: 1485 super: 453	83%	ordinary: 1287 super: 219
$n=10$	11%	ordinary: 35 super: 31	27%	ordinary: 385 super: 280	44%	ordinary: 1925 super: 1085	59%	ordinary: 5005 super: 2065	72%	ordinary: 7007 super: 1967
$n=11$	9%	ordinary: 44 super: 40	22%	ordinary: 616 super: 479	36%	ordinary: 4004 super: 2545	50%	ordinary: 14014 super: 6989	63%	ordinary: 28028 super: 10483
$n=12$	7%	ordinary: 54 super: 50	18%	ordinary: 936 super: 763	31%	ordinary: 7644 super: 5285	43%	ordinary: 34398 super: 19537	55%	ordinary: 91728 super: 41447
$n=13$	6%	ordinary: 65 super: 61	16%	ordinary: 1365 super: 1152	26%	ordinary: 13650 super: 10038	38%	ordinary: 76440 super: 47712	48%	ordinary: 259896 super: 134316
$n=14$	5%	ordinary: 77 super: 73	13%	ordinary: 1925 super: 1668	23%	ordinary: 23100 super: 17802	33%	ordinary: 157080 super: 105288	43%	ordinary: 659736 super: 376908

Table 1. Comparison of the *number of diagrams* in the ordinary MHV vertex expansion and the super MHV vertex expansion for n-point N^kMHV amplitudes in the range $n = 5, \ldots, 14$, $k = 1, \ldots, 5$. The *percentage of eliminated diagrams* is also displayed. (Amplitudes which are anti-N^qMHV with $q < k$ and thus more efficiently computed using an anti-MHV vertex expansion are displayed in gray.)

Using these analytic expressions for $M(n,k)$ and $S(n,k)$, the elimination ratio $1 - S/M$, which is also displayed in table 1, can be easily analyzed. While the percentage of eliminated diagrams decreases with n at fixed k, the elimination ratio increases along the diagonal, when k and n are increased simultaneously. Generically, the computationally most challenging amplitudes are gluon amplitudes with an equal number of negative and positive helicity legs (and amplitudes related to these by supersymmetry). For these $2m$-point $N^{(m-2)}$MHV amplitudes, the elimination ratio remains non-zero even for $m \to \infty$, approaching $11/27 \approx 41\%$ in this limit.

Pure gluon amplitudes. If we consider the computation of a particular $\mathcal{N} = 4$ SYM amplitude, not every diagram contributes to its ordinary MHV vertex expansion. In fact, there are diagrams for which no assignment of states to the internal lines can turn all subamplitudes in the diagram into MHV vertices. A simple example is the 6-gluon amplitude $\mathcal{A}_6^{\text{NMHV}}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$, to which the diagrams $\alpha = \{4,5,6\}$, $\{4,5\}$, and $\{5,6\}$ do not contribute, because the subamplitude containing the three negative helicity gluon lines $1, 2, 3$ cannot be MHV. More generally, each SU(4) index a imposes constraints on the possible diagrams that contribute, and forces certain diagrams to vanish. For pure-gluon amplitudes, all SU(4) indices appear on the same lines, and they thus all impose the same constraints. Therefore, many more MHV vertex diagrams contribute to pure-gluon amplitudes than to generic $\mathcal{N} = 4$ SYM amplitudes.

In the super MHV vertex expansion, we choose the η_{Xa} to eliminate diagrams. Such

a choice, however, can cause diagrams to reappear that would have been absent in the ordinary MHV vertex expansion. We witnessed this phenomenon in the example of the 5-point NMHV amplitude $\mathcal{A}_5^{\mathrm{NMHV}}(1^+, 2^+, 3^-, 4^-, 5^-)$ discussed in section 3.2. There, the only diagram contributing to the super MHV vertex expansion was $\alpha = \{1, 2\}$, precisely the diagram that would not have contributed to the ordinary MHV vertex expansion.

If we are interested in one particular pure-gluon amplitude, and many diagrams for that amplitude already vanish in its ordinary MHV vertex expansion, it is advisable to pick only three channels $\beta_1, \beta_2, \beta_3$, and to set $\eta_{X4} = 0$. The constraints from the SU(4) index $a = 4$ then still enforce the vanishing of diagrams absent in its ordinary MHV vertex expansion.

As an example, consider the 8-point $\mathrm{N}^2\mathrm{MHV}$ amplitude $\mathcal{A}_8^{\mathrm{N}^2\mathrm{MHV}}(1^-, 2^-, 3^-, 4^-, 5^+, 6^+, 7^+, 8^+)$. Forty-four out of 120 diagrams contribute to its ordinary MHV vertex expansion. We can eliminate 22 further diagrams by choosing $\beta_1 = \{3, 4\}$, $\beta_2 = \{3, 4, 5\}$, and $\beta_3 = \{4, 5\}$. The super MHV vertex expansion of this amplitude thus only contains half as many non-vanishing diagrams as the ordinary MHV vertex expansion. Although we could only use three of the β_a to eliminate channels, this elimination ratio of 50% is even better than the generic ratio of 44% given in table 1 for the 8-point $\mathrm{N}^2\mathrm{MHV}$ level. Despite its supersymmetric origin, the super MHV vertex expansion is thus no less powerful when applied to QCD amplitudes.

4 All-line supershifts

In the next two sections, we will show that the super MHV vertex expansion presented above follows naturally from the recursion relation associated with holomorphic all-line supershifts. In the current section, we motivate and define holomorphic all-line supershifts, and study the behavior of generating functions under these supershifts.

Supershifts were introduced in ref. [14–16] as a generalization of an ordinary two-line shift in the BCFW approach [3]. An ordinary BCFW shift $[p, q\rangle$ is defined as

$$|p] \;\to\; |p] + z|q], \qquad\qquad |q\rangle \;\to\; |q\rangle - z|p\rangle, \qquad\qquad (4.1)$$

with all other angle and square spinors remaining unshifted. Under such a shift of spinors, the scattering amplitude acquires a dependence on z. If the shift is such that the deformed amplitude $\mathcal{A}(z)$ vanishes as $z \to \infty$, then the shift gives rise to a valid BCFW recursion relation for the amplitude. In ref. [3], it was shown that the validity of a shift depends on the helicities of the lines p and q participating in the shift. Consequently, the generating functions, which encode amplitudes of all helicities, will not generally vanish at large z under a BCFW shift, though the coefficients of some of its η-monomials will.

A *BCFW supershift* $[p, q\rangle$ is a generalization of eq. (4.1) that acts on Grassmann variables as well [13–16]:

$$|p] \to |p] + z|q], \qquad |q\rangle \to |q\rangle - z|p\rangle, \qquad \eta_{pa} \;\to\; \eta_{pa} + z\eta_{qa}, \qquad (4.2)$$

with all other angle spinors, square spinors, and Grassmann variables η_{ia} remaining unshifted. The advantage of the supershift is that $\sum_{i=1}^n |i\rangle \eta_{ia}$ remains invariant, which improves the large z falloff of generating functions under this shift. For example, the MHV

generating function

$$\mathcal{F}_n^{\text{MHV}}(\eta_{ia}) = \frac{\delta^{(8)}\left(\sum_{i=1}^n |i\rangle \eta_{ia}\right)}{\text{cyc}(1, \ldots, n)} \tag{4.3}$$

vanishes at least as $1/z$ for large z, for any choice of lines p and q (it goes as $1/z^2$ if p and q are not adjacent). This behavior generalizes beyond the MHV level. Indeed, all generating functions $\mathcal{F}_n^{N^k \text{MHV}}(\eta_{ia})$ vanish at large z for any choice of lines p and q and hence may be represented by *super BCFW recursion relations* [13–16].

The MHV vertex expansion of an amplitude, on the other hand, may be derived from a holomorphic shift, i.e. a shift that acts on square spinors only and leaves all angle spinors invariant. A holomorphic all-line shift was defined in ref. [6] as

$$|i] \to |i] + z\, c_i |X], \qquad\qquad |i\rangle \to |i\rangle \qquad\qquad \text{for} \quad i = 1, \ldots, n \;, \tag{4.4}$$

where $|X]$ is an arbitrary reference spinor, and the complex parameters c_i are constrained by momentum conservation to obey

$$\sum_{i=1}^n c_i |i\rangle = 0 \,. \tag{4.5}$$

We also demand that the sum of momenta be unchanged under an all-line supershift *only* when *all* external momenta are summed. Specifically, we require

$$\sum_{i \in \alpha} c_i |i\rangle \neq 0 \tag{4.6}$$

for all proper subsets α of consecutive external lines. The MHV generating function (4.3) is manifestly invariant under this shift. Furthermore, it was shown in ref. [6] that N^kMHV generating functions $\mathcal{F}_n^{N^k \text{MHV}}(\eta_{ia})$, and thus all N^kMHV amplitudes, fall off at least as $1/z^k$ under an all-line shift. Therefore all amplitudes in $\mathcal{N} = 4$ SYM theory may be represented by the MHV vertex expansion.

We generalize the shift (4.4) to a *holomorphic all-line supershift*

$$|i] \to |i] + z\, c_i |X], \qquad |i\rangle \to |i\rangle, \qquad \eta_{ia} \;\to\; \eta_{ia} + z\, c_i \eta_{Xa} \qquad \text{for} \quad i = 1, \ldots, n \;, \tag{4.7}$$

where, in addition to the reference spinor $|X]$, we introduce four arbitrary reference Grassmann parameters η_{Xa}. We still impose the conditions (4.5) and (4.6). The former in particular implies that $\delta(\sum_{i=1}^n |i\rangle \eta_{ia})$ is invariant under the supershift (4.7) since

$$\sum_{i=1}^n |i\rangle \eta_{ia} \;\to\; \sum_{i=1}^n |i\rangle \eta_{ia} + z \sum_{i=1}^n c_i |i\rangle \eta_{Xa} = \sum_{i=1}^n |i\rangle \eta_{ia} \,. \tag{4.8}$$

Consequently, the MHV generating function (4.3) is invariant under a holomorphic all-line supershift. In section 4.1, we study the behavior of anti-MHV generating functions under this shift. In section 4.2, we will show that the N^kMHV generating function falls off at least as $1/z^k$ under an all-line supershift, and in section 5, we will use its recursion relation to derive the super MHV vertex expansion.

4.1 Anti-MHV generating functions

We now consider the behavior of anti-MHV generating functions under supershifts. Due to their simplicity, anti-MHV generating functions are the ideal testing ground to study the large-z falloff of generating functions under supershifts. We will examine this falloff for both generic holomorphic all-line supershifts and a particularly interesting restricted class of such supershifts.

The n-point anti-MHV generating function, expressed in terms of the conjugate Grassmann variables $\bar{\eta}_i^a$, is given by

$$\overline{\mathcal{F}}_n^{\overline{\mathrm{MHV}}}(\bar{\eta}_i^a) = \frac{\delta^{(8)}\left(\sum_{i=1}^n |i]\bar{\eta}_i^a\right)}{\prod_{i=1}^n [i, i+1]} \,. \tag{4.9}$$

The numerator of the anti-MHV generating function,

$$\delta^{(8)}\left(\sum_{i=1}^n |i]\bar{\eta}_i^a\right) = \frac{1}{2^4} \prod_{a=1}^4 \sum_{i,j=1}^n [i\,j]\,\bar{\eta}_i^a\,\bar{\eta}_j^a \,, \tag{4.10}$$

may be recast as a function of η_{ia}:

$$\delta^{(8)}\left(\sum_{i=1}^n |i]\bar{\eta}_i^a\right) \xrightarrow{\mathrm{GFT}} \frac{\prod_{a=1}^4 \sum_{j_1,\ldots,j_n=1}^n \epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2]\,\eta_{j_3 a} \cdots \eta_{j_n a}}{[2\,(n-2)!]^4} \,. \tag{4.11}$$

Here, we used the Grassmann Fourier transform (GFT) [16, 19]:

$$\bar{f}(\bar{\eta}_i^a) \xrightarrow{\mathrm{GFT}} f(\eta_{ia}) \equiv \int \prod_{i,a} d\bar{\eta}_i^a \, \exp\left(\sum_{b,j} \eta_{jb}\bar{\eta}_j^b\right) \bar{f}(\bar{\eta}_i^a) \,. \tag{4.12}$$

Hence the anti-MHV generating function is given by [5]

$$\mathcal{F}_n^{\overline{\mathrm{MHV}}}(\eta_{ia}) = \frac{\prod_{a=1}^4 \sum_{j_1,\ldots,j_n=1}^n \epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2]\,\eta_{j_3 a} \cdots \eta_{j_n a}}{[2\,(n-2)!]^4 \prod_{i=1}^n [i, i+1]} \,. \tag{4.13}$$

The sum over $\epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2]\,\eta_{j_3 a} \cdots \eta_{j_n a}$ is invariant under a BCFW supershift (4.2). Therefore the anti-MHV generating function, just as the MHV-generating function, falls off as $1/z$ (or $1/z^2$) under a BCFW supershift.

Under a holomorphic all-line supershift (4.7), the sum over $\epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2]\,\eta_{j_3 a} \cdots \eta_{j_n a}$ generically picks up a piece linear in z; higher powers of z cancel due to the antisymmetry of $\epsilon^{j_1 j_2 \cdots j_n}$. Since each square bracket in the denominator of eq. (4.13) generically goes as z, the anti-MHV generating function vanishes as $1/z^{n-4}$ for large z under a holomorphic all-line supershift (4.7). Since an n-point anti-MHV amplitude is an $\mathrm{N}^k\mathrm{MHV}$ amplitude with $k = n - 4$, we conclude that

$$\mathcal{F}_n^{\overline{\mathrm{MHV}}}(\eta_{ia}) \sim \frac{1}{z^k} \tag{4.14}$$

under a generic holomorphic all-line supershift.

While the falloff (4.14) is sufficient to derive all-line supershift recursion relations for anti-MHV generating functions with $n > 4$ external legs, it is worthwhile to examine whether the $1/z^k$ falloff can be improved for a careful choice of shift parameters. This possibility is interesting for two reasons. First of all, improved large z behavior implies the presence of additional sum rules. Secondly, a faster falloff could justify a super MHV vertex expansion for generating functions in $\mathcal{N} = 8$ supergravity. The simplicity of the anti-MHV generating functions allows us to explicitly test the possibility of improved large z behavior.

As we will now see, the anti-MHV generating function indeed falls off faster than $1/z^{n-4}$ under a restricted class of holomorphic all-line supershifts. Expand the square spinor $|X]$ and the Grassmann parameters η_{Xa} in (4.7) as

$$|X] = \sum_{i=1}^{n} d_i |i], \qquad \eta_{Xa} = \sum_{i=1}^{n} d_i \eta_{ia} . \qquad (4.15)$$

Note that we use the same expansion coefficients d_i for both $|X]$ and η_{Xa}, which is a very special non-generic choice. Under such a supershift, one may show that

$$\sum_{j_1,\ldots,j_n=1}^{n} \epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2] \, \eta_{j_3 a} \cdots \eta_{j_n a} \to \left(1 + z \sum_{i=1}^{n} c_i d_i \right) \sum_{j_1,\ldots,j_n=1}^{n} \epsilon^{j_1 j_2 \cdots j_n} [j_1 j_2] \, \eta_{j_3 a} \cdots \eta_{j_n a} . \qquad (4.16)$$

We now further restrict the parameters d_i in eq. (4.15) to satisfy

$$\sum_{i=1}^{n} c_i d_i = 0 , \qquad (4.17)$$

making the numerator of eq. (4.13) *invariant* under the supershift. Since each square bracket in the denominator of eq. (4.13) goes as z (for $n > 4$), the anti-MHV generating function vanishes as $1/z^n$ at large z. The generic $1/z^k$ behavior under a holomorphic all-line supershift is thus improved, and we have

$$\mathcal{F}_n^{\overline{\mathrm{MHV}}}(\eta_{ia}) \ \sim \ \frac{1}{z^{k+4}} \quad \text{for} \ \ n > 4, \quad \text{when} \ \ \sum_{i=1}^{n} c_i d_i = 0 . \qquad (4.18)$$

The significance of this result is that it implies the validity of a super MHV vertex expansion for anti-MHV generating functions in $\mathcal{N} = 8$ supergravity, as we will argue in section 6.

The improved falloff (4.18) does not hold for $n = 4$ external legs. As anti-MHV four-point functions are also MHV, they must be invariant under any holomorphic all-line supershift and cannot possibly go as $1/z^4$. In fact, the kinematics of four-point functions ensures that the square brackets in the denominator of eq. (4.13) are invariant under supershifts that satisfy the condition (4.17). The generating function for four-point anti-MHV amplitudes is thus indeed invariant under such a supershift.

It is instructive to derive the result (4.18) in a different way by considering the anti-MHV generating function in the form (4.9). When condition (4.17) holds, the linear shift on η_{ia} is equivalent to a linear shift on $\bar{\eta}_i^a$ in the Fourier-transformed generating function

$\overline{\mathcal{F}}(\bar{\eta}_i^a)$. In fact, for any functions $f(\eta_{ia})$ and $\bar{f}(\bar{\eta}_i^a)$ which are related by a Grassmann Fourier transformation, one has

$$
f\big(\eta_{ia} + zc_i\textstyle\sum_j d_j\eta_{ja}\big) \xrightarrow{\text{GFT}} \int\prod_{i,a} d\eta_{ia}\exp\Big(\sum_{b,j}\bar{\eta}_j^b\eta_{jb}\Big) f\Big(\eta_{ia} + zc_i\textstyle\sum_j d_j\eta_{ja}\Big)
$$
$$
= \int\prod_{i,a} d\eta'_{ia}\exp\Big(\sum_{b,j}\Big[\bar{\eta}_j^b - z\,d_j\textstyle\sum_k c_k\bar{\eta}_k^b\Big]\,\eta'_{jb}\Big) f\big(\eta'_{ia}\big)
$$
$$
= \bar{f}\Big(\bar{\eta}_i^a - z\,d_i\textstyle\sum_j c_j\bar{\eta}_j^a\Big)\,. \tag{4.19}
$$

Here, the condition (4.17) was necessary to show that the change of variables $\eta'_{ia} = \eta_{ia} + zc_i\sum_j d_j\eta_{ja}$ implies $\eta_{ia} = \eta'_{ia} - zc_i\sum_j d_j\eta'_{ja}$ and to guarantee that the measure is invariant under the change of variables. We conclude that the conjugate Grassmann variables $\bar{\eta}_i^a$ transform linearly under a supershift:

$$
\eta_{ia} \to \eta_{ia} + z\,c_i\sum_{j=1}^n d_j\eta_{ja} \iff \bar{\eta}_i^a \to \bar{\eta}_i^a - z\,d_i\sum_{j=1}^n c_j\bar{\eta}_j^a \qquad \text{when} \qquad \sum_{i=1}^n c_i d_i = 0\,. \tag{4.20}
$$

Note that the roles of c_i and d_i are reversed in the shifts of η_{ia} and $\bar{\eta}_i^a$.

Since by eq. (4.15) we have

$$
|i] \to |i] + z\,c_i\sum_{j=1}^n d_j|j]\,, \tag{4.21}
$$

it follows that $\sum_{i=1}^n |i]\bar{\eta}_i^a$ and therefore $\delta^{(8)}\big(\sum_{i=1}^n |i]\bar{\eta}_i^a\big)$ is invariant under this restricted class of supershifts.[10] The improved $1/z^n$ falloff is then manifest in the anti-MHV generating function (4.9).

4.2 N^kMHV generating functions under all-line supershifts

We now show that generating functions for N^kMHV amplitudes fall off at least as $1/z^k$ for large z under all-line supershifts with shift parameters $|X]$ and η_{Xa}. For $k \geq 1$ these supershifts thus give valid recursion relations. As we will show in the following section, the associated recursion relations imply the super MHV vertex expansion. Our derivation of the falloff is based on the ordinary MHV vertex expansion of the generating function, whose validity was established in refs. [5, 6]. An alternative derivation, based on the super BCFW recursion relations of ref. [13–16] is outlined in appendix A.1.

Consider the behavior of each of the terms in the ordinary MHV vertex expansion (3.29) under a generic holomorphic all-line supershift (4.7). Crucially, we use a reference spinor $|Z]$ in the ordinary MHV vertex expansion that does not coincide with the spinor $|X]$

---

[10]If we impose the even stronger condition $c_i d_i = 0$ for each i, then the holomorphic all-line supershift is manifestly a composition of several $[p,q\rangle$ supershifts (4.2) and the invariance of $\sum_{i=1}^n |i]\bar{\eta}_i^a$ automatically follows. Note, though, that such a shift is no longer an all-line shift, as we need to set at least one $c_i = 0$ to satisfy this stronger condition.

appearing the supershift (4.7). We demand $[XZ] \neq 0$. The shift then acts on the CSW spinors $|P_{\alpha_A}\rangle$ in an MHV vertex diagram as

$$|P_{\alpha_A}\rangle = P_{\alpha_A}|Z] \;\rightarrow\; \hat{P}_{\alpha_A}|Z] = P_{\alpha_A}|Z] + z \sum_{j \in \alpha_A} c_j |j\rangle [XZ]. \qquad (4.22)$$

Condition (4.6) ensures that the $O(z)$ term on the right hand side does not vanish. For any external line i, and any two internal lines P_{α_A} and P_{α_B} of an MHV vertex diagram, we then find the following shift dependence:

$$\langle i \hat{P}_{\alpha_A}\rangle \sim z, \qquad\qquad \langle \hat{P}_{\alpha_A} \hat{P}_{\alpha_B}\rangle \sim z^2. \qquad (4.23)$$

We can thus associate one power of z with each occurrence of $|\hat{P}_{\alpha_A}\rangle$. As the CSW spinor $|\hat{P}_{\alpha_A}\rangle$ of each internal line α_A appears four times in the cyclic factors of the denominator of (3.29), we have

$$\frac{1}{\text{cyc}(\hat{I}_1)\cdots\text{cyc}(\hat{I}_{k+1})} \;\sim\; \frac{1}{z^{4k}}. \qquad (4.24)$$

The spin factors in the numerator of (3.29) shift as

$$\sum_{i \in \alpha_A} \langle i \hat{P}_{\alpha_A}\rangle \hat{\eta}_{ia} \;=\; z^2 \sum_{i,j \in \alpha_A} c_i c_j \langle ij\rangle [XZ]\eta_{Xa} + O(z) \;=\; O(z), \qquad (4.25)$$

thus overall the numerator goes at most as z^{4k}. Taking into account the shift dependence of each of the k propagators, $1/\hat{P}_{\alpha_A}^2 \sim 1/z$, we conclude that the MHV vertex expansion of the generating function, diagram by diagram, falls off at least as

$$\mathcal{F}_n^{\text{N}^k\text{MHV}}(\eta_{ia}) \;\sim\; \frac{1}{z^k} \qquad (4.26)$$

under holomorphic all-line supershifts.

5 The super MHV vertex expansion from all-line supershifts

In this section we derive the super MHV vertex expansion from holomorphic all-line supershifts. The proof is a generalization of the derivation of the ordinary MHV vertex expansion in ref. [6]. We will thus emphasize the new aspects of the current proof, referring the reader to the details in ref. [6] for steps that proceed analogously.

5.1 All-line supershift recursion relations

We proved in section 4 that N^kMHV generating functions $\mathcal{F}_n^{\text{N}^k\text{MHV}}(\eta_{ia})$ with $k \geq 1$ vanish as $z \rightarrow \infty$ under an all-line supershift. All-line supershifts can thus be used to derive a recursion relation for $\mathcal{F}_n^{\text{N}^k\text{MHV}}(\eta_{ia})$. Each diagram in the recursion relation is the product of two generating functions, connected by a scalar propagator $1/P_\alpha^2$. We denote the set of external states on one subamplitude by α, and their associated Grassmann variables by $\{\eta_{ia}\}_{i \in \alpha}$. Similarly, on the other subamplitude we denote the external states and associated Grassmann variables by $\bar{\alpha}$ and $\{\eta_{ia}\}_{i \in \bar{\alpha}}$, respectively. The generating functions of the two subamplitudes also depend on the Grassmann variable $\eta_{P_\alpha a}$ associated with the internal

propagator line. In the recursion relation we need to carry out the intermediate state sum by acting with the Grassmann differential operator $D_\alpha^{(4)} = \prod_a \partial/\partial\eta_{P_\alpha a}$. We thus obtain

$$
\mathcal{F}_n^{\mathrm{N}^k\mathrm{MHV}} \tag{5.1}
$$
$$
= \frac{1}{2} \sum_{q=0}^{k-1} \sum_\alpha D_\alpha^{(4)} \frac{\mathcal{F}^{\mathrm{N}^q\mathrm{MHV}}\big(\hat{\alpha}, -\hat{P}_\alpha\,; \{\hat{\eta}_{ia}\}_{i\in\alpha}\,, \eta_{P_\alpha a}\big)\, \mathcal{F}^{\mathrm{N}^{(k-q-1)}\mathrm{MHV}}\big(\hat{\bar{\alpha}}, \hat{P}_\alpha\,; \{\hat{\eta}_{ia}\}_{i\in\bar{\alpha}}\,, \eta_{P_\alpha a}\big)}{P_\alpha^2}\bigg|_{z=z_\alpha}.
$$

The notation $\hat{\alpha}$, $\hat{\bar{\alpha}}$, $\hat{P}_\alpha, \hat{\eta}_{ia}$ indicates that the momenta and Grassmann variables of the subamplitudes are shifted. They are evaluated at $z = z_\alpha$ satisfying the pole condition $\hat{P}_\alpha^2(z) = 0$, i.e.

$$
z_\alpha = \frac{P_\alpha^2}{\sum_{i\in\alpha} c_i \langle i|P_\alpha|X]}. \tag{5.2}
$$

Condition (4.6) ensures that z_α is always well-defined, and thus all possible diagrams α contribute to the recursion relation. Since $\hat{P}_\alpha$ is null when $z = z_\alpha$, we can write

$$
\hat{P}_\alpha = |\hat{P}_\alpha\rangle[\hat{P}_\alpha| \implies |\hat{P}_\alpha\rangle = \frac{\hat{P}_\alpha|X]}{[\hat{P}_\alpha X]} = \frac{P_\alpha|X]}{[\hat{P}_\alpha X]}. \tag{5.3}
$$

Finally, the symmetry factor $\frac{1}{2}$ in eq. (5.1) is necessary because, for each channel α, we now also include the equivalent term with $\alpha \leftrightarrow \bar{\alpha}$ in the sum.

5.2 NMHV generating function

At the NMHV level, all subamplitudes in the recursion relation (5.1) are MHV. Using the MHV generating function (2.3) we obtain

$$
\mathcal{F}_n^{\mathrm{NMHV}} = \sum_{\text{diagrams } \alpha} D_\alpha^{(4)} \frac{\delta^{(8)}\big(\sum_{i\in\alpha} |i\rangle\hat{\eta}_{ia} - |\hat{P}_\alpha\rangle\eta_{P_\alpha a}\big)\, \delta^{(8)}\big(\sum_{i\in\bar{\alpha}} |i\rangle\hat{\eta}_{ia} + |\hat{P}_\alpha\rangle\eta_{P_\alpha a}\big)}{\mathrm{cyc}(\hat{I}_1)\, P_\alpha^2\, \mathrm{cyc}(\hat{I}_2)}\bigg|_{z=z_\alpha}. \tag{5.4}
$$

where $|\hat{P}_\alpha\rangle$ in the numerator, and in the cyclic factors in the denominator, is given by eq. (5.3). Using $\delta^{(8)}(A)\delta^{(8)}(B) = \delta^{(8)}(A)\delta^{(8)}(A+B)$ and the invariance of $\sum_{i=1}^n |i\rangle\eta_{ia}$ from eq. (4.8), we obtain

$$
\mathcal{F}_n^{\mathrm{NMHV}} = \sum_{\text{diagrams } \alpha} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\, P_\alpha^2\, \mathrm{cyc}(I_2)} \prod_{a=1}^4 \sum_{i\in\alpha} \langle iP_\alpha\rangle\hat{\eta}_{ia}\bigg|_{z=z_\alpha}. \tag{5.5}
$$

Since all factors of $[\hat{P}_\alpha X]$ cancel, we used the CSW prescription

$$
|\hat{P}_\alpha\rangle \rightarrow P_\alpha|X] = |P_\alpha\rangle \tag{5.6}
$$

for all occurrences of $|\hat{P}_\alpha\rangle$ in the numerator and the cyclic factors in the denominator of eq. (5.5).

Let us now examine the effect of the shifted $\hat{\eta}_{ia}$ in eq. (5.5). We find

$$
\sum_{i\in\alpha} \langle iP_\alpha\rangle\hat{\eta}_{ia}\bigg|_{z=z_\alpha} = \sum_{i\in\alpha} \langle iP_\alpha\rangle\eta_{ia} + z_\alpha \sum_{i\in\alpha} c_i\langle iP_\alpha\rangle\eta_{Xa} = \sum_{i\in\alpha} \langle iP_\alpha\rangle\eta_{ia} + P_\alpha^2\,\eta_{Xa}, \tag{5.7}
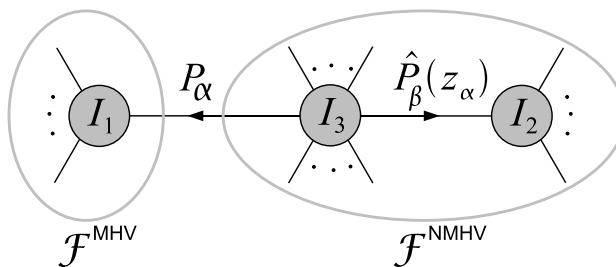$$

Figure 3. A diagram α of the all-line supershift recursion relation of an N^{2}MHV generating function. The super MHV vertex expansion is substituted for the NMHV subamplitude, and the internal line P_β of the super MHV vertex diagram is evaluated at shifted momenta.

where we inserted z_α from eq. (5.2) in the last step. We obtain

$$\mathcal{F}_n^{\text{NMHV}} = \sum_{\text{diagrams }\alpha} \frac{\delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_{ia}\right)}{\text{cyc}(I_1)\, P_\alpha^2\, \text{cyc}(I_2)}\; \prod_{a=1}^4 \left[\sum_{i\in\alpha}\langle iP_\alpha\rangle\eta_{ia} + P_\alpha^2\,\eta_{Xa}\right].\qquad (5.8)$$

We have thus reproduced $\tilde{\mathcal{F}}_n^{\text{NMHV}}(\eta_{ia};\eta_{Xa})$ given in eq. (3.11), i.e. the form of the NMHV generating function associated with the super MHV vertex expansion.

5.3 N^{2}MHV generating function

Let us now use eq. (5.1) to determine the generating function at the N^{2}MHV level. One subamplitude is again MHV, for which we use the MHV generating function in the form (2.3). For the NMHV subamplitude, it is crucial that we use the generating function (5.8) associated with the super MHV vertex expansion, and that we use the same reference parameters $|X]$, η_{Xa} in the generating function as in the all-line supershift. The NMHV generating function contains a sum over channels which we denote by β. The propagator line P_α is an "external" line of the NMHV subamplitude, but we choose β to *not* include this line ($P_\alpha \notin \beta$), so that both α and β only contain external lines of the full amplitude (see figure 3).

A short calculation analogous to that in ref. [6] gives

$$\mathcal{F}_n^{\text{N}^2\text{MHV}} = \sum_{\alpha,\beta} \frac{\delta^{(8)}\left(\sum_{i=1}^n |i\rangle\eta_{ia}\right)}{\text{cyc}(I_1)\text{cyc}(I_2)\text{cyc}(I_3)}\, \frac{1}{P_\alpha^2\, \hat{P}_\beta^2(z_\alpha)} \prod_{a=1}^4 \left[\sum_{i\in\alpha}\langle i\,P_\alpha\rangle\hat{\eta}_{ia}\right]\left[\sum_{i\in\beta}\langle i\,P_\beta\rangle\hat{\eta}_{ia} + \hat{P}_\beta^2(z_\alpha)\,\eta_{Xa}\right].$$
$$(5.9)$$

Note that the angle brackets $\langle i\,P_\beta\rangle$ (including those implicit in $\text{cyc}(I_2)$ and $\text{cyc}(I_3)$) are unaffected by the shift because

$$|\hat{P}_\beta(z_\alpha)\rangle = \hat{P}_\beta(z_\alpha)|X] = P_\beta|X] + z_\alpha\sum_{i\in\beta}c_i|i\rangle[XX] = P_\beta|X] = |P_\beta\rangle.\qquad (5.10)$$

To simplify the last factor in eq. (5.9), we employ the crucial identity

$$\sum_{i\in\beta}\langle i\,P_\beta\rangle\hat{\eta}_{ia} + \hat{P}_\beta^2(z_\alpha)\eta_{Xa} = \sum_{i\in\beta}\langle i\,P_\beta\rangle\eta_{ia} + P_\beta^2\,\eta_{Xa}\,,\qquad (5.11)$$

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where we used

$$z_\alpha \sum_{i\in\beta} \langle i\, P_\beta \rangle + \hat{P}_\beta^2(z_\alpha) = P_\beta^2 \,. \tag{5.12}$$

Note that the identity (5.11) relies on the fact that the all-line supershift recursion relation and the NMHV generating function for the subamplitude are based on the same reference parameters $|X]$ and η_{Xa}. We can then rewrite eq. (5.9) as

$$\mathcal{F}_n^{\mathrm{N^2MHV}} = \sum_{\alpha,\beta} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\mathrm{cyc}(I_2)\mathrm{cyc}(I_3)} \frac{1}{P_\alpha^2 \hat{P}_\beta^2(z_\alpha)} \prod_{a=1}^4 \left[\sum_{i\in\alpha} \langle i\, P_\alpha\rangle\eta_{ia} + P_\alpha^2\,\eta_{Xa}\right]\left[\sum_{i\in\beta} \langle i\, P_\beta\rangle\eta_{ia} + P_\beta^2\,\eta_{Xa}\right]. \tag{5.13}$$

Symmetrizing the sum in $\alpha \leftrightarrow \beta$ and using the identity [21]

$$\frac{1}{P_\alpha^2 \hat{P}_\beta^2(z_\alpha)} + \frac{1}{\hat{P}_\alpha^2(z_\beta)\, P_\beta^2} = \frac{1}{P_\alpha^2 P_\beta^2} \,, \tag{5.14}$$

we find

$$\mathcal{F}_n^{\mathrm{N^2MHV}} = \frac{1}{2}\sum_{\alpha,\beta} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\mathrm{cyc}(I_2)\mathrm{cyc}(I_3)} \frac{1}{P_\alpha^2 P_\beta^2} \prod_{a=1}^4 \left[\sum_{i\in\alpha} \langle iP_\alpha\rangle\eta_{ia} + P_\alpha^2\eta_{Xa}\right]\left[\sum_{i\in\beta} \langle iP_\beta\rangle\eta_{ia} + P_\beta^2\eta_{Xa}\right]. \tag{5.15}$$

As there was no restriction on our original sum over α, we are counting each distinct MHV vertex diagram twice in eq. (5.15). The factor of $\frac{1}{2}$ compensates this overcounting. We then have (after relabeling $\alpha \to \alpha_1$ and $\beta \to \alpha_2$)

$$\mathcal{F}_n^{\mathrm{N^2MHV}} = \sum_{\substack{\mathrm{MHV\ diagrams}\\ \{\alpha_1,\alpha_2\}}} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\mathrm{cyc}(I_2)\mathrm{cyc}(I_3)} \prod_{A=1}^2 \frac{1}{P_{\alpha_A}^2} \prod_{a=1}^4 \left[\sum_{i\in\alpha_A} \langle i\, P_{\alpha_A}\rangle\eta_{ia} + P_{\alpha_A}^2\eta_{Xa}\right]. \tag{5.16}$$

We have thus derived the super MHV vertex expansion (3.31) at the $\mathrm{N^2MHV}$ level.

5.4 All tree amplitudes

The generalization to the $\mathrm{N}^k\mathrm{MHV}$ generating function with $k \geq 3$ is straightforward. We continue inductively in k, plugging in the super MHV vertex expansion for the $\mathrm{N}^q\mathrm{MHV}$ subamplitudes ($q < k$) in the recursion relation (5.1). We use the same reference parameters $|X]$, η_{Xa} in the subamplitudes as in the all-line supershift recursion relation. The proof proceeds precisely as in ref. [6], except that we need to use the identity (5.11) for all channels $\beta_1, \dots, \beta_q$ appearing in an $\mathrm{N}^q\mathrm{MHV}$ subamplitude. The only remaining shift dependence then resides in the propagators, and using the identity [6, 21]

$$\sum_{A=1}^k \frac{1}{\hat{P}_{\alpha_1}^2(z_{\alpha_A})\cdots \hat{P}_{\alpha_{A-1}}^2(z_{\alpha_A})\, P_{\alpha_A}^2\, \hat{P}_{\alpha_{A+1}}^2(z_{\alpha_A})\cdots \hat{P}_{\alpha_k}^2(z_{\alpha_A})} = \frac{1}{P_{\alpha_1}^2 \cdots P_{\alpha_k}^2} \,, \tag{5.17}$$

we obtain the generating function

$$\mathcal{F}_n^{\mathrm{NMHV}}(\eta_{ia}) = \sum_{\substack{\mathrm{MHV\ diagrams}\\ \{\alpha_1,\dots,\alpha_k\}}} \frac{\delta^{(8)}\big(\sum_{i=1}^n |i\rangle\eta_{ia}\big)}{\mathrm{cyc}(I_1)\cdots\mathrm{cyc}(I_{k+1})} \prod_{A=1}^k \frac{1}{P_{\alpha_A}^2} \prod_{a=1}^4 \left[\sum_{i\in\alpha_A} \langle iP_{\alpha_A}\rangle\eta_{ia} + P_{\alpha_A}^2\eta_{Xa}\right]. \tag{5.18}$$

This generating function coincides with the super MHV vertex expansion (3.31), and completes our derivation.

As a consistency check, one can show that the super MHV vertex expansion immediately implies the $1/z^k$ falloff of N^kMHV generating functions under all-line supershifts. This check is carried out in appendix A.2.

6 Discussion

In this paper we have presented a new family of representations for the generating functions of $\mathcal{N} = 4$ SYM theory, the super MHV vertex expansion. The diagrams of this family of representations depend on a reference spinor $|X]$ and on four reference Grassmann parameters η_{Xa}, which may be chosen arbitrarily. We have shown that the super MHV vertex expansion arises both from a particular supersymmetry transformation on the ordinary MHV vertex expansion, and from the recursion relations associated with holomorphic all-line supershifts. This family of shifts similarly depends on reference parameters $|X]$ and η_{Xa}, which then results in the dependence of the super MHV vertex expansion on these parameters. The ordinary MHV vertex expansion corresponds to the special case $\eta_{Xa} = 0$, but certain non-trivial choices for η_{Xa} can significantly reduce the number of diagrams contributing to the expansion and thus simplify the task of computing amplitudes.

The efficient computation of on-shell tree amplitudes of $\mathcal{N} = 4$ SYM theory has various applications. For example, tree amplitudes are an important ingredient for the computation of loop amplitudes using (generalized) unitarity cuts [22–31]. The scattering amplitudes of $\mathcal{N} = 4$ SYM theory are of particular interest, as the AdS/CFT correspondence permits insights into their strong coupling behavior (see e.g. [32] and references therein).

Another application is the computation of on-shell tree amplitudes of $\mathcal{N} = 8$ supergravity, which can be expressed in terms of $\mathcal{N} = 4$ SYM amplitudes through the KLT relations [33]. They then also play an important role in the study of $\mathcal{N} = 8$ supergravity at loop level [34]. Loop amplitudes in $\mathcal{N} = 8$ supergravity exhibit surprising properties [16, 35–38], and their UV behavior has recently been under intense investigation due to the possible perturbative finiteness of the theory [31, 34, 39–45].

KLT relations can be used to relate not only the on-shell tree amplitudes of $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ SYM, but their generating functions as well. Very recently, this was carried out in [46] using the generating functions [13] based on dual superconformal symmetry [12, 15, 19, 47–57], and using the KLT relations in the form of ref. [58]. The final expression for the resulting $\mathcal{N} = 8$ supergravity generating function contained far fewer terms than naively expected [46]. It would be interesting to see whether a similar simplification occurs when the generating function of the $\mathcal{N} = 4$ (super) MHV vertex expansion is used to determine an $\mathcal{N} = 8$ supergravity generating function via KLT.

It would also be interesting to find generating functions for $\mathcal{N} = 8$ supergravity amplitudes directly from $\mathcal{N} = 8$ recursion relations. To derive recursion relations, one needs to determine shifts (or supershifts) under which an amplitude (or generating function) vanishes as the deformation parameter z is taken to infinity. $\mathcal{N} = 8$ supergravity amplitudes generically do not vanish under holomorphic shifts when the number n of external lines becomes large. In fact, under holomorphic shifts, amplitudes go as $z^{n-\ell}$ for some integer ℓ.

For example, pure-graviton NMHV amplitudes go as z^{n-12} under a holomorphic shift of the three negative helicity graviton lines [10, 59]. The MHV vertex expansion for gravity [21] is then not valid for graviton NMHV amplitudes with $n \geq 12$ external lines. The falloff becomes even worse for more general external states, and the MHV vertex expansion in $\mathcal{N} = 8$ supergravity has not even been established for general 5-point NMHV amplitudes, and has been shown to fail for certain scalar amplitudes at the 6-point level.

In this paper, we found that holomorphic all-line supershifts with suitably chosen shift parameters yield $1/z^{k+4}$ suppression for anti-MHV amplitudes in $\mathcal{N} = 4$ SYM theory. This immediately implies, via the KLT relations, that $\mathcal{N} = 8$ supergravity anti-MHV generating functions with $n > 4$ external legs go at least as $z^{n-11-2k} = z^{-n-3}$ under a suitable holomorphic all-line supershift. A valid recursion relation, namely the super MHV vertex expansion for $\mathcal{N} = 8$ supergravity, can thus be derived at the anti-MHV level.[11] It would be interesting to determine the precise form of this expansion, and to study the sum rules implied by η_{Xa} independence.[12] It would be particularly interesting to see whether the improved falloff of $1/z^{k+4}$ in $\mathcal{N} = 4$ SYM theory, and thus the improved falloff of at least $z^{n-11-2k}$ in $\mathcal{N} = 8$ supergravity, can be generalized beyond the anti-MHV level. If so, the validity of a super MHV vertex expansion for $\mathcal{N} = 8$ supergravity could also be extended beyond the anti-MHV level.

In this paper, we presented the super MHV vertex expansion as an on-shell recursion relation associated with a complex shift. Various off-shell approaches, however, have also proved useful to gain insights into the ordinary MHV vertex expansion [61–70]. It would be interesting to see whether the super MHV vertex expansion has a natural interpretation in an off-shell framework.

Acknowledgments

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[11]To see this, observe that all-line supershift recursion relations express an n-point anti-MHV generating function purely in terms of lower-point anti-MHV generating functions and 3- or 4-point MHV generating functions. A super MHV vertex expansion for arbitrary anti-MHV generating functions in $\mathcal{N} = 8$ supergravity can thus be established inductively in n.

[12]Other interesting sum rules for $\mathcal{N} = 8$ supergravity were recently studied in [60].

A Large-z falloff under holomorphic all-line supershifts

A.1 $\mathcal{F}_n^{N^k MHV} \sim 1/z^k$ using the super BCFW recursion relations

In this appendix we outline the derivation of the $1/z^k$ falloff of N^kMHV generating functions under holomorphic all-line supershifts using the supersymmetric generalization of the BCFW recursion relation [13–16]. This generalizes the derivation of ref. [6] from ordinary all-line shifts to all-line supershifts.

The $1/z^k$ falloff of the N^kMHV generating function $\mathcal{F}_n^{N^kMHV}(\eta_{ia})$ under an ordinary all-line shift was derived in ref. [6] by recursively studying the behavior of the BCFW representation of N^kMHV amplitudes. The inductive argument presented there relied on three facts:

1. Each $\mathcal{N} = 4$ SYM amplitude admits at least one valid BCFW recursion relation [5, 71, 72].

2. An all-line shift on an amplitude acts, to leading order in z, as an all-line shift on the subamplitudes of each diagram in its BCFW representation.

3. The falloff of $1/z^k$ is valid for all MHV and anti-MHV amplitudes.

As a generalization of this argument, we now use the super BCFW recursion relations to study the behavior of generating functions under all-line supershifts. It suffices to repeat steps (1)–(3) for this case, and we will now briefly outline how this is done.

As shown in ref. [16], all generating functions vanish at large z under a super BCFW shift (4.2) for any choice of two lines p and q, and thus admit a valid super BCFW recursion relation. This establishes the analog of (1).

For (2), we need to show that an all-line supershift on the entire generating function acts, to leading order in z, as an all-line supershift on the subamplitudes of each diagram in the super BCFW representation. Notice that the kinematics are unaltered compared to the ordinary all-line shift, and the analysis of ref. [6] thus establishes that the angle and square brackets of the super BCFW subamplitudes are subject to a holomorphic all-line supershift. In particular, it follows from ref. [6] that the square spinor associated with the internal line of momentum P_α in a $[p, q\rangle$ super BCFW recursion relation shifts as

$$|P_\alpha] \quad \rightarrow \quad z c_{P_\alpha} |X] + \mathcal{O}(1), \qquad \text{with} \qquad c_{P_\alpha} = \sum_{i \in \alpha} c_i \frac{\langle pi \rangle}{\langle pq \rangle} \tag{A.1}$$

under the all-line supershift. It remains to analyze the dependence of the super BCFW subamplitudes on the Grassmann variables. We recall from eq. (4.8) that all-line supershifts are designed to leave the argument of the overall $\delta^{(8)}$ in the generating function invariant. For any super BCFW diagram characterized by an internal line of momentum P_α, it is easy to show that the change in the argument of the $\delta^{(8)}$ of either of its subamplitudes is $O(1)$ under the all-line supershift provided that the Grassmann variable associated with the internal line undergoes the shift

$$\eta_{P_\alpha a} \quad \rightarrow \quad \eta_{P_\alpha a} + z \, c_{P_\alpha} \eta_{Xa} \,, \tag{A.2}$$

where c_{P_α} is defined in eq. (A.1). The Grassmann variables $\eta_{P_\alpha a}$ are integrated over to carry out the intermediate state sum in the super BCFW diagram. Therefore, the shift (A.2) can be implemented as a change of variables in the Grassmann integral over $\eta_{P_\alpha a}$. After this change of variables, an all-line supershift on the whole amplitude manifestly acts, to leading order, as an all-line supershift on the super BCFW subamplitudes. This establishes the analog of (2).

The analog of (3), i.e.the $1/z^k$ behavior of MHV and anti-MHV generating functions under all-line supershifts, was established in section 4.1. The inductive argument of ref. [6] for ordinary all-line shifts thus carries over to the case of supershifts.

A.2 $\mathcal{F}_n^{N^k MHV} \sim 1/z^k$ using the super MHV vertex expansion

In this paper, we have seen the super MHV vertex expansion emerge from two different approaches: first, in section 3, as a supersymmetry transformation of the ordinary MHV vertex expansion, and second, in section 5, as the recursion relation implied by all-line supershifts. In this appendix we close the loop and show that the super MHV vertex expansion immediately implies the $1/z^k$ falloff of N^{k}MHV generating functions under all-line supershifts. This serves as another consistency check on our result.

Consider the action of an all-line supershift (4.7) on the n-point N^{k}MHV generating function. We choose to represent this generating function as the super MHV vertex expansion (5.18) with the reference parameters $|X]$ and η_{Xa} chosen to coincide with those of the supershift. Inspecting eq. (5.18), we find that all k propagators shift as $1/\hat{P}_{\alpha_A}^2 \sim 1/z$, giving $1/z^k$ suppression. The cyclic factors in the denominator are invariant because the CSW spinors $|P_{\alpha_A}\rangle$ are invariant by eq. (5.10). The spin factors $\sum_{i \in \alpha_A} \langle i\, P_{\alpha_A} \rangle \eta_{ia} + P_{\alpha_A}^2 \eta_{Xa}$ in the numerator are also invariant, by the identity (5.11). We thus find $1/z^k$ suppression diagram-by-diagram in the super MHV vertex expansion, and conclude that N^{k}MHV generating functions fall off at least as $1/z^k$ under all-line supershifts. As we saw in section 4.1, the falloff can be even stronger for certain amplitudes and shifts, but unfortunately the super MHV vertex expansion is not sensitive to this stronger falloff, which arises through cancellations between diagrams.

This argument was, not surprisingly, simpler than the derivations of the $1/z^k$ falloff from the ordinary MHV vertex expansion and the super BCFW recursion relation that we presented in section 4.2 and appendix A.1, respectively. Just as the $1/z^k$ falloff under ordinary all-line shifts was naturally shown from the ordinary MHV vertex expansion with coinciding reference spinor in ref. [6], we have now done the same for all-line supershifts using the super MHV vertex expansion with coinciding reference spinor and coinciding reference Grassmann parameters.

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